Green Correspondence

Let G be a finite group, and K a field with char $K = p \mid |G|$

If $A = KG$, $A = \oplus P_i$ where each P_i proj. indecomp.

Thm 5.3: There is a one-to-one corresp.

$$\left\{ \begin{array}{l} \text{Iso classes of indecomp} \\ \text{proj. } A\text{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Iso classes of} \\ \text{simple } A\text{-mods} \end{array} \right\}$$

by associating $S_i = P_i / \text{rad} P_i$ to each indecomp. proj P_i

Ex: $G = \mathbb{Z}/p\mathbb{Z}$, $K = \overline{\mathbb{F}}_p$, then $KG \cong K[x]/x^p$

As a module KG_{KG} indecomp. The radical is $(x)/x^p$, so the only simple module is K.

→ The full list of indecomps is $\{ K[x]/x^n \mid 1 \le n \le p \}$

Ex: $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, $K = \overline{\mathbb{F}}_p$. Then $KG = K[x,y]/x^p, y^p$

KG_{KG} indecomp w/ radical (x,y), so only one simple K.

→ For each $n \in \mathbb{Z}_{>0}$ let V_n have $\dim V_n = 2n$ w/ basis $v_1, \dots, v_n, w_1, \dots, w_n$. Let $X, Y \in \text{End}_K(V_n)$ s.t.

$$X v_i = w_i \quad X w_i = 0$$
$$Y v_i = w_{i+1} \quad Y w_j = 0$$

$$\begin{tikzcd} v_1 \arrow[d, "x"] \arrow[dr, "y"] & v_2 \arrow[d, "x"] \arrow[dr, "y"] & \cdots & v_n \arrow[d, "x"] \\ w_1 & w_2 & \cdots & w_n \end{tikzcd}$$

Let $x \in KG$ act by $(I+X)$ and $y \in KG$ act by $(I+Y)$.

This is an infinite list of indecomp.

How do we begin to classify indecomps?

→ Locally

Prop 5.1: A KG-module U is free, if and only if, U has a subspace $\mathcal{X}$ s.t. any linear transformation from $\mathcal{X}$ to any KG-module V extends uniquely to a module hom U to V.

Proof:

$\Rightarrow$) Pick a basis of $U = \bigoplus_{i=1}^{n} KG$, $\{e_1, \dots, e_n\}$ and let $\mathcal{X} = \langle e_i \rangle_{i=1}^{n}$

If $\varphi : \mathcal{X} \longrightarrow V$, define $\Phi(u) = \Phi(\sum a_i e_i) = \sum a_i \varphi(e_i)$

$\Leftarrow$) If W has subspace Y of dim n, then the vector space iso $\psi : \mathcal{X} \rightarrow Y$ has an inverse & both of which extend uniquely to isos $\Psi : U \longrightarrow W$, $\Phi : W \longrightarrow U$, so $W \cong U$.

Remark: A subspace is just a $\{1\}$-module, so why not extend to other subgroups.

$$\begin{tikzcd} & U_G \arrow[dr, "\Phi_{KG}"] & \\ \mathcal{X}_H \arrow[ur] \arrow[rr, "\varphi_{KH}"'] & & V_G \end{tikzcd}$$

Def: A KG-module is relatively H-free, $H \leq G$, if there is a KH-module $\mathcal{X}$ of U s.t. any KH-hom from $\mathcal{X}$ to a KG module V extends uniquely to a KG-hom $U \rightarrow V$

Prop 8.2: If $\mathcal{X}$ a KH-module, where $H \leq G$, then there is a KG-module which is relatively H-free wrt $\mathcal{X}$.

Proof: Let S be left coset reps of H, then each $\{s\} \times \mathcal{X}$ is a vector space and $U = \bigoplus_{s \in S} \{s\} \times \mathcal{X}$ (all sums $\sum_{s \in S} (s, x_s)$).

It has KG-module structure given by $\forall g \in G, s \in S, x \in \mathcal{X}$:

$$g \cdot (s, x) = (t, hx) \quad \text{where } gs = th,\ t \in S,\ h \in H$$

Remark: The map $\varphi : V^G = V \otimes_{KH} KG \longrightarrow U$ makes V^G a relatively H-free module wrt V.

Notation: V^G the induced module, V_H the restriction.

Prop 9.1: Let U a KG-module and $H \leq G$, TFAE

(1) U a direct summand of a rel. H-free module

(2) If $\varphi: V \twoheadrightarrow U$ a KG-hom and φ splits as a KH-hom, then φ is split $(0 \to \ker\varphi \to U \to P \to 0)$

(3) If $\varphi: V \twoheadrightarrow W$ a KG-hom and $\psi: U \to W$, then there is a KG-hom $\rho: U \to V$ s.t. $\varphi\rho = \psi$ provided there is a KH-hom w/ this prop

(4) U is a direct summand of $(U_H)^G$

Def: Such modules are called rel. H-proj.

Remark: Since V^G rel. H-free for a KH-module V, a summand U is relatively H-projective.

Ex: $G = D_3 = \langle r, t \rangle$, $H = \langle t \rangle \cong \mathbb{Z}/2\mathbb{Z}$ $K = \overline{\mathbb{F}}_2$ $V = K[x]/x^2$

$S = \{e, r, r^2\}$

$U = \langle (e,1), (e,x), (r,1), (r,x), (r^2,1), (r^2,x) \rangle \cong KG_{KG}$

rewrite U w/ basis $\langle (e,1), (e,x+1), \ldots \rangle$

Then $U_H \cong \bigoplus_{i=1}^{3} K\mathbb{Z}/2\mathbb{Z}$

So $(U_H)^G = \bigoplus^3 KH \otimes_{KH} KG_{KG} \cong \bigoplus^3 KG_{KG}$

Thus, U a summand of $(U_H)^G$.

Notation: • $U \mid V$ means U is iso to a direct summand of V

Thm 9.4: Let U be an indecomp. KG-module

(1) There is a p-subgroup Q of G, unique up to conjugacy in G, s.t. U is rel. H-projective, for $H \leq G$ iff H contains a conj of Q.

(2) There is an idecomp. KQ-module S, unique up to conjugacy in $N_G(Q)$, s.t. $U \mid S^G$.

Def: Q is a <u>vertex</u> of U, S a <u>source</u> of U.

Remark: Q "measures" how close U is to being projective.

if U projective it is $\{1\}$-proj. and so $Q \leq \{1\}$ is trivial (1)

if $Q = \{1\}$, S^G is free and $U | S^G$ (2).

Notation: G a finite group

- Q a p-subgroup of G
- L a subgroup containing $N_G(Q)$
- $H, R \leq G$, $H \leq_G R$ when R contains a conj. of H.
- If $\mathcal{H}$ a collection of subgroups, $P \leq_G \mathcal{H}$ means $P \leq_G H$ for some $H \in \mathcal{H}$

$\mathfrak{X} = \{sQs^{-1} \cap Q \mid s \in G \setminus L\}$

$\mathfrak{Y} = \{sQs^{-1} \cap L \mid s \in G \setminus L\}$

$\mathfrak{Z} = \{R \mid R \leq G, R \not\leq_G \mathfrak{X}\}$

we say a KG-module is rel. $\mathcal{H}$-proj. if U is a direct sum of modules, each of which is rel H-proj for $H \in \mathcal{H}$.

Thm 11.1: There is a 1-1 corresp.

$$\left\{\begin{array}{l}\text{Iso classes of indecomp.}\\ KG\text{-modules w/ vertex in } \mathfrak{Z}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Iso classes of indecomp.}\\ KL\text{-modules w/ vertex in } \mathfrak{Z}\end{array}\right\}$$

If U and V are such modules for G and L, resp. which corresp. then U and V have the same vertex and

$$U_L \cong V \oplus Y \qquad V^G \cong U \oplus X$$

where Y is a rel. $\mathfrak{Y}$-proj KL-module and X a rel. $\mathfrak{X}$-proj KG-module.

Lemma 9.5: If U is indecomp. KG-module w/ vertex Q and $G \geq H \geq Q$ then there exists an indecomp. KH-module V satisfying any two of the following

(1) $V \mid U_H$ (2) $U \mid V^G$ (3) V has vertex Q.

Proof:

Since U has vertex Q, it is rel. H-proj and so $U \mid (U_H)^G$

Let V be an indecomp. summand of U_H s.t. $U \mid V^G$

$\rightarrow U_H = \bigoplus T_i \Rightarrow (U_H)^G = \bigoplus (T_i \otimes_{KH} KG) \Rightarrow U \mid T_j \otimes_{KH} KG$ choose $V = T_j$

Then $V \mid U_H^{(1)}$ and $U \mid V^{G\,(2)}$

Let S be a source for U (so KQ-mod). Then $U \mid S^G$.

Since $S^G \cong (S^H)^G$, choose V an indecomp summand of S^H

$S^H = \bigoplus T_i \otimes_{KQ} KH \Rightarrow (S^H)^G = \bigoplus T_i \otimes_{KQ} KG$, $U \mid T_j \otimes_{KQ} KG$

choose $V = T_j \otimes_{KQ} KH \Rightarrow U \mid V^G$

So $U \mid V^G$ (2)

Claim: V has vertex Q

by thm 9.4

Since $V \mid S^H$, V is rel. Q-proj. Then $R \leq Q$ is a vertex of V.

Now, let W be a KR-module s.t. $V \mid W^H$ (W a source) so

$V^G \mid (W^H)^G \cong W^G$

But since $U \mid V^G$, $U \mid W^G$ and so U rel R-proj.

Because U has vertex Q, and is rel. R-proj. R contains a conj. of Q.

But $R \leq Q$, so $R = Q$ and V has vertex Q. (3)

Since $U \mid (U_Q)^G$ ($\mathrm{Hom}_G(U, \mathrm{Ind}(\mathrm{res}\, U)) \cong \mathrm{Hom}_Q(\mathrm{res}\, U, \mathrm{res}\, U) \ni id$)

it is Q-proj. and so there is $S \mid U_Q$ s.t. $U \mid S^G$.

Let V be an indecomp summand $V \mid U_H$, so $V_Q \mid (U_H)_Q \cong U_Q$ s.t. $S \mid V_Q$.

Claim: V has vertex Q.

$V \mid U_H \mid (S^G)_H$. By Mackey's $V \mid ({}^s S)_{H \cap sQs^{-1}})^H$ for some $s \in G$.

__Lemma 9.6:__ If V is a rel. Q-proj. kH-module, where $Q \leq H$
Then $(V^G)_H \cong V \oplus W$ where every indecomp. summand of W is relatively proj. for a subgroup of the form $sQs^{-1} \cap H$, $s \in G$, $s \notin H$.

__Recall Mackey's Thm:__ $(V^G)_L \cong \bigoplus_{s \in L\backslash G/H} (s(V)_{L \cap sHs^{-1}})^L$ where $s(V) \cong s \otimes V$ and $L\backslash G/H$ are double cosets.
$$L\backslash G/H = \{LgH\} \qquad LgH = \{lgh \mid l \in L, h \in H\}$$

__Proof:__ Since V rel. Q-proj, V has source U s.t. $V \mid U^H$. Then $U^H \cong V \oplus T$ for some kH-mod T.

Then $U^G \cong V^G \oplus T^G$ and $(U^G)_H \cong V \oplus W \oplus T \oplus X$
where $(V^G)_H \cong \underline{V \oplus W}$ and $(T^G)_H \cong T \oplus X$ for kH-modules W and X.

By Mackey's thm $(U^G)_H \cong \bigoplus_{s \in H\backslash G/Q} (s(U)_{H \cap sQs^{-1}})^H \cong U^H \oplus Y$
where the summand w/ $s \in H$ gives U^H and Y is everything else.
Since we induced up from groups $H \cap sQs^{-1}$ each sumand of Y is rel. proj. for subgroups of the form $sQs^{-1} \cap H$, $s \notin H$. So $\underline{W \oplus X \cong Y}$.

__cont:__
Then V has vertex $R \leq H \cap sQs^{-1}$. Then $V \mid W^H$ for some kR-mod W.
Since $s \mid V_Q$, $s \mid (W^H)_Q$.
By Mackey's $(W^H)_Q \cong \bigoplus_{h \in Q\backslash H/R} (s(W)_{Q \cap hRh^{-1}})^Q$, so S is relatively $Q \cap hRh^{-1}$-proj.
Because $U \mid S^G$, S has vertex Q, otherwise U would be and have vertex smaller than Q.
Thus $Q \subseteq hRh^{-1}$, but $R \leq sQs^{-1}$, so $Q = hRh^{-1}$ and V has vertex Q.

Lemma 11.2: If R is a subgroup of Q then TFAE

(1) $R \leq_G \ast$ (2) $R \leq_L \ast$ (3) $R \leq_L \sim$

Proof:

Lemma 11.3: If U is rel. $*$-proj KG-module then U_L is rel. η-proj. If V is a rel. η-proj KL-module then V^G is rel. $*$-proj.

Proof:

Lemma 11.5: If V an indecomp. KL-module w/ vertex R in $\mathfrak{Z}$ then $V^G \simeq U \oplus X$ where U is an indecomp. KG-module w/ vertex R, $V | U_L$ and X is rel. $\mathfrak{X}$-proj. KG-module.

Proof: Let $V^G = \bigoplus_{i=1}^{r} U_i$ w/ each U_i indecomp. KG-modules.

By lemma 9.6, $(V^G)_L \simeq V \oplus Y$ where Y a rel. η-proj. KL-module.

Renumber so that $(U_1)_L \simeq V \oplus Y_1$ and $(U_i)_L \simeq Y_i$ $i \geq 2$ where Y_i are KL-modules and $Y \simeq Y_1 \oplus \dots \oplus Y_r$.

Claim: U_1 has vertex in $\mathfrak{Z}$ and $U_2, \dots, U_r$ are rel. $\mathfrak{X}$-proj.

Since $U_1 | V^G$, U_1 has vertex $R \leq Q$

If U_1 was $\mathfrak{X}$-proj. then by lemma 11.3 $(U_1)_L \simeq V \oplus Y_1$ would be η-proj but V has vertex $Q \notin \eta$, so it's not the case. So U_1 has vertex in $\mathfrak{Z}$

if have vertex R, then $U | (U_R)^G$, so U is R-proj

If U_i not rel. $\mathfrak{X}$-proj. then by lemma 11.4 $(U_i)_L$ would not be η-proj. but it is. Thus U_i are $\mathfrak{X}$-proj.

Set $U = U_1$, $X = U_2 \oplus \dots \oplus U_r$ we get $U | V^G$ and $V^G \simeq U \oplus X$ where X rel. $\mathfrak{X}$-proj.

Claim: U has vertex R.

Since U has vertex in $\mathfrak{Z}$, by lemma 11.4, $U_L \simeq V \oplus Y$, where V is not η-proj. Moreover V has the same vertex as U, so both are R

We need only show the 1-1 property, that is,

if U an indecomp KG-module w/ vertex $R \in \mathfrak{Z}$

- V is as in Lemma 11.4
- $V^G = U' \oplus Y$ as in Lemma 11.5

then $U \simeq U'$. Similarly for V.

But in lemma 11.4 we showed $U | V^G$ and in 11.5 $V | U_L$, so we are done

Lemma 11.4: If U is an indecomp. kG-module w/ vertex $R \in \mathfrak{Z}$ then $U_L \cong V \oplus Y$, where V an indecomp kL-module w/ vertex R, $U | V^G$ and Y is a relatively η-proj. kL-module.

Proof: By lemma 9.5 ($H=L, Q=R$) there is an indecomp. kL-module V w/ vertex R and $U | V^G$.

Now by lemma 9.6 ($Q=R, H=L$) $(V^G)_L \cong V \oplus Y$, where Y_1 rel. η-proj.

Then U_L is either iso to $V \oplus Y$ or Y, where Y a summand of Y_1.

By lemma 9.5, applied to an indecomp. summand of U_L, call it W, W has vertex R

$\boxed{\text{By lemma 11.2}}$ if W is iso to a summand of Y, then $R \subseteq_L \eta$ since Y_1 is η-proj. and so $R \in_G \mathfrak{X}$. But this means $R \notin \mathfrak{Z}$ which is a contra. to assump.

by def of $\mathfrak{Z}$

Thus $W \cong V$ and $U_L \cong V \oplus Y$.